COTANGENT BUNDLES OF TORIC VARIETIES AND COVERINGS OF TORIC HYPERKÄHLER MANIFOLDS

CRAIG VAN COEVERING AND WEI ZHANG

ABSTRACT. Toric hyperkähler manifolds are quaternion analog of toric varieties. Bielawski pointed out that they can be glued by cotangent bundles of toric varieties. Following his idea, viewing both toric varieties and toric hyperkäher manifolds as GIT quotients, we first establish geometrical criteria for the semi-stable points. Then based on these criteria, we show that the cotangent bundles of compact toric varieties in the core of toric hyperkähler manifold are sufficient to glue the desired toric hyperkähler manifold.

1. INTRODUCTION

Toric varieties are originally defined by the combinatorial data of the fans(cf. [Ful93] or [Oda88]), also studied by Delzant and Guillemin ([Del88], [Gui94b]) from symplectic quotient perspective, which has natural connection with the Geometric Invariant Theory(cf. [MFK94]). Toric hyperkähler manifold is a quaternion analogue of toric variety which carries hyperkähler metric automatically. In [BD00], Bielawski and Dancer studied their basic properties: moment map, core, cohomology, etc. In [Bie99], Bielawski pointed out that toric hyperkähler manifold can be viewed as gluing the cotangent bundles of toric varieties, however, he didn't give the explicit way of gluing.

Following his instruction, we pursue a "canonical" set of toric varieties whose cotangent bundles are enough to glue the toric hyperkähler manifold. For this purpose, we combine the symplectic and GIT quotients methods. In section 2, we give the definitions and basic properties of toric varieties and toric hyperkähler manifolds as symplectic quotients.

The essential new ingredient of this paper is in section 3. In [Kon03], Konno present the GIT quotient construction of toric hyperkähler manifold and give a numerical criterion for the semi-stable points. Basing on his method, we derive a similar criterion for the toric varieties. Then we establish a geometric interpretations of these two semi-stability criteria, which are the affine analogs of the state sets in projective case due to Dolgachev and Hu in [DH98]. Namely, for a toric variety $X(\alpha)$ with hyperplanes arrangement $\mathcal{A} = \{(H_i, u_i)\}_{i=1}^{d}$, where $H_i = \{x \in \mathfrak{n}^* | \langle u_i, x \rangle + \lambda_i = 0\}$, we can set the half space

$$H_i^{\geq 0} = \{x \in \mathfrak{n}^* | \langle u_i, x \rangle + \lambda_i \geq 0\},$$

and for every $z \in \mathbb{C}^d$ define $St_{\mathcal{A}}(z) = \bigcap_{i=1}^{d} F_z(i)$, where

$$F_z(i) = \begin{cases} H_i^{\geq 0} & if\, z_i \neq 0 \\ H_i & if\, z_i = 0 \end{cases}.$$

Thus viewing toric variety $X(\alpha)$ as a GIT quotient, we have

Proposition 1.1. *A point $z \in \mathbb{C}^d$ is α-semi-stable if and only if the set $St_{\mathcal{A}}(z) \subset \mathfrak{n}^*$ is nonempty.*

Similarly, for a toric hyperkähler variety $Y(\alpha, \beta)$(see the detailed definition $St_{\mathcal{A}}(z, w)$ in section 3), we have

Proposition 1.2. *A point $(z, w) \in \mu_{\mathbb{C}}^{-1}(\beta)$ is α-semi-stable if and only if $St_{\mathcal{A}}(z, w) \subset \mathfrak{n}^*$ is nonempty.*

Above two criteria can be viewed as the dual version about the hyperplane arrangement in $\mathfrak{n}^*$ of Konno's numerical ones about the moment map value α in $\mathfrak{m}^*$. The most important feature of ours is that they will enable us to analyze the structure of toric hyperkähler manifolds in a efficient way.

In section 4, after recalling the basic relationship between toric varieties and toric hyperkähler manifolds, namely the extended core and the core of toric hyperkähler manifold, and cotangent bundles of toric varieties. Denoting the open sets identical to the cotangent bundles of compact toric varieties in the core of toric hyperkähler manifold as $U_\epsilon, \epsilon \in \Theta_{cpt}$, we prove the main theorem,

Theorem 1.3. *Let $Y(\alpha, 0)$ be a smooth toric hyperkähler manifold with nonempty core, then the canonical open set $U_\epsilon \cong T^*X(\mathcal{A}_\epsilon), \epsilon \in \Theta_{cpt}$ is a covering of $Y(\alpha, 0)$, i.e. the cotangent bundles of compact toric varieties in the core are enough to glue $Y(\alpha, 0)$.*

Finally, there are a Corollary asserting that each individual $T^*X(\mathcal{A}_\epsilon)$ is dense in $Y(\alpha,0)$ and a Conjecture that its complement is of complex dimension n and constituted by several toric varieties intersecting together.

2. PRELIMINARY

2.1. **Toric variety.** We first state the symplectic definition about toric varieties. The real torus $T^d = \{(\zeta_1, \zeta_2, \cdots, \zeta_d) \in \mathbb{C}^d, |\zeta_i| = 1\}$ acts on $\mathbb{C}^d$ freely. Denote M the m-dimensional connected subtorus of T^d whose Lie algebra $\mathfrak{m} \subset \mathfrak{t}^d$ is generated by integer vectors(which we shall always take to be primitive), then we have the following exact sequence

$$0 \to \mathfrak{m} \xrightarrow{\iota} \mathfrak{t}^d \xrightarrow{\pi} \mathfrak{n} \to 0,$$

$$0 \leftarrow \mathfrak{m}^* \xleftarrow{\iota^*} (\mathfrak{t}^d)^* \xleftarrow{\pi^*} \mathfrak{n}^* \leftarrow 0,$$

where $\mathfrak{n} = \mathfrak{t}^d/\mathfrak{m}$ is the Lie algebra of the n-dimensional quotient torus $N = T^d/M$ and $m + n = d$. For simplicity, we omit the superscript d over $\mathfrak{t}$ from now on.

Let $\{e_i\}_{i=1}^d$ be the standard basis of $\mathfrak{t}$, then $\pi(e_i) = u_i$ are also primitive. Denote $\{e_i^*\}_{i=1}^d$ the dual basis of $\mathfrak{t}^*$ and $\{\theta_i\}_{i=1}^m$ some basis span $\mathfrak{m}$. The action of M on $\mathbb{C}^d$ admits a moment map

$$\mu(z) = \frac{1}{2} \sum_{i=1}^d |z_i|^2 \iota^* e_i^*.$$

For any $\alpha \in \mathfrak{m}^*$, the symplectic quotient $\mu^{-1}(\alpha)/M$ is a toric variety, denoted as $X(\alpha)$, inheriting Kahler metric from $\mathbb{C}^d$ on it's smooth part(cf. [Gui94a]). The quotient torus N has a residue circle action on $X(\alpha)$ and gives rise to a moment map to $\mathfrak{n}^*$,

$$\bar{\mu}([z]) = \frac{1}{2} \sum_{i=1}^d |z_i|^2 e_i^*.$$

The image of this map is a convex polytope Δ called the Delzant polytope of $X(\alpha)$(cf. [Del88]).

Conversely, any smooth compact toric variety X of complex dimension n, with a Kahler metric invariant under some torus N comes from Delzant's construction. Unfortunately, this polytope does not recover all the data of the quotient construction, and the worse is that it does

not cooperate well with the toric hyperkähler theory. We use the notion of hyperplanes arrangement with orientation(cf. [Pro04]) to replace polytope. In detail, consider a set of rational oriented hyperplanes $\mathcal{A} = \{(H_i, u_i)\}_{i=1}^d$,

$$H_i = \{x \in \mathfrak{n}^* | \langle u_i, x \rangle + \lambda_i = 0\},$$

where H_i is the hyperplane and u_i is fixed primitive vector in $\mathfrak{n}_\mathbb{Z}$ specifying the orientation, called the normal of H_i. We define several subspaces related to these oriented hyperplanes,

$$H_i^{\geq 0} = \{x \in \mathfrak{n}^* | \langle u_i, x \rangle + \lambda_i \geq 0\},$$
$$H_i^{\leq 0} = \{x \in \mathfrak{n}^* | \langle u_i, x \rangle + \lambda_i \leq 0\}.$$

A ploytope is naturally associated to this arrangement,

$$\Delta = \bigcap_{i=1}^d H_i^{\geq 0},$$

which could be empty or unbounded.

Then the arrangement $\mathcal{A}$ will decide a toric variety the same as Δ does. Since u_i define a map $\pi : \mathfrak{t} \to \mathfrak{n}$, where $\mathrm{Ker}\pi = \mathfrak{m}$, let M be the Lie group corresponding to $\mathfrak{m}$ and set $\alpha = \sum \lambda_i \iota^* e_i^*$, then we call $\mu^{-1}(\alpha)/M$ the toric variety corresponding to $\mathcal{A}$ and $\lambda = (\lambda_1, \cdots, \lambda_d)$ a lift of α. For fixed normal vectors, the hyperplane arrangements corresponding to two different lifts of same moment map value α only differ by a parallel transport, thus produce same toric variety(cf. [Pro04]). So we can abuse the notations $X(\alpha)$ and $X(\mathcal{A})$.

Example 2.1 (see [BD00] or [Pro04]). *Let $n = 2$, $u_1 = f_1$, $u_2 = f_2$, $u_3 = -f_1 - f_2$, $u_4 = -f_2$, and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$. The toric variety is Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O})$. See Figure 1.*

If the rational vectors u_i is regular, i.e. every collection of n linearly independent vectors $\{u_{i_1}, \cdots, u_{i_n}\}$ span $\mathfrak{n}_\mathbb{Z}$ as a $\mathbb{Z}$-basis, then $\mathcal{A}$ is called regular. The arrangement $\mathcal{A}$ is called simple if every subset of k hyperplanes with nonempty intersection intersects in codimension k. Then $\mathcal{A}$ is smooth if it is both regular and simple. It is not difficult to see that $X(\mathcal{A})$ is smooth if and only if $\mathcal{A}$ is smooth.

From now on, $\mathcal{A}$ is always assumed to be smooth. If we denote the regular value set of moment map μ as $\mathfrak{m}_{reg}^*$, it is easy to check this condition is equivalent to $\{u_i\}$ is regular and $\alpha \in \mathfrak{m}_{reg}^*$. Moreover, denote Θ the set of maps form $\{1, \ldots, d\}$ to $\{-1, 1\}$. For $\epsilon \in \Theta$, let $\mathcal{A}_\epsilon$ be the arrangement changing the normal of H_i if $\epsilon(i) = -1$, and when $\epsilon(i) = 1$ for all i, we abbreviate the subscript ϵ for simplicity. Notice that the toric variety $X(\mathcal{A}_\epsilon)$ for various ϵ could be totally different.

(a) the fan

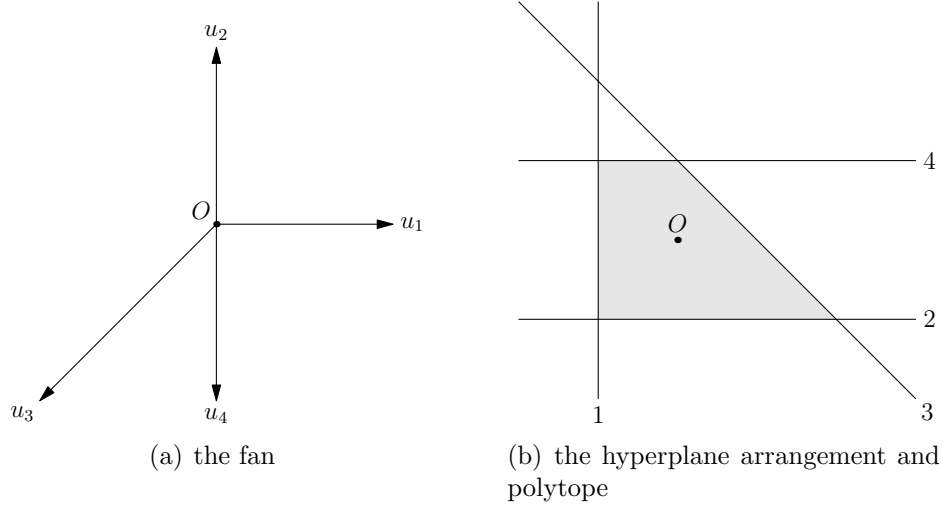(b) the hyperplane arrangement and polytope

FIGURE 1. the fan and arrangement in Example 2.1

2.2. Toric hyperkähler manifold.

A $4n$-dimensional manifold is hyperkähler if it possesses a Riemannian metric g which is Kahler with respect to three complex structures I_1; I_2; I_3 satisfying the quaternionic relations $I_1 I_2 = -I_2 I_1 = I_3$ etc. To date the most powerful technique for constructing such manifolds is the hyperkähler quotient method of Hitchin, Karlhede, Lindstrom and Rocek([HKLR87]). We specialized on the class of hyperkähler quotients of flat quaternionic space $\mathbb{H}^d$ by subtori of T^d. The geometry of these spaces turns out to be closely connected with the theory of toric varieties.

Since $\mathbb{H}^d$ can be identified with $T^* \mathbb{C}^d \cong \mathbb{C}^d \times \mathbb{C}^d$, it has three complex structures $\{I_1, I_2, I_3\}$. The real torus $T^d = \{(\zeta_1, \zeta_2, \cdots, \zeta_d) \in \mathbb{C}^d, |\zeta_i| = 1\}$ acts on $\mathbb{C}^d$ induce a action on $T^* \mathbb{C}^d$ keeping the hyperkähler structure,

$$(z, w)\zeta = (z\zeta, w\zeta^{-1}).$$

The subtours M acts on it admitting a hyperkähler moment map $\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) : \mathbb{H}^d \to \mathfrak{m}^* \times \mathfrak{m}_{\mathbb{C}}^*$, given by

$$\mu_{\mathbb{R}}(z, w) = \frac{1}{2} \sum_{i=1}^{d} (|z_i|^2 - |w_i|^2) \iota^* e_i^*,$$

$$\mu_{\mathbb{C}}(z, w) = \sum_{i=1}^{d} z_i w_i \iota^* e_i^*,$$

where the complex moment map $\mu_{\mathbb{C}} : \mathbb{H}^d \to \mathfrak{m}_{\mathbb{C}}^*$ is holomorphic with respect to I_1. Bielawski and Dancer introduced the definition of toric

hyperkähler varieties, and generally speaking, they are not toric varieties.

Definition 2.2 ([BD00]). *A toric hyperkähler variety $Y(\alpha, \beta)$ is a hyperkähler quotient $\mu^{-1}(\alpha, \beta)/M$ for $(\alpha, \beta) \in \mathfrak{m}^* \times \mathfrak{m}_{\mathbb{C}}^*$.*

A smooth part of $Y(\alpha, \beta)$ is a $4n$-dimensional hyperkähler manifold, whose hyperkähler structure is denoted by (g, I_1, I_2, I_3). The quotient torus $N = T/M$ acts on $Y(\alpha, \beta)$, preserving its hyperkähler structure. This residue circle action admits a hyperkähler moment map $\bar{\mu} = (\bar{\mu}_{\mathbb{R}}, \bar{\mu}_{\mathbb{C}})$,

$$\bar{\mu}_{\mathbb{R}}([z, w]) = \frac{1}{2} \sum_{i=1}^{d} (|z_i|^2 - |w_i|^2) e_i^*,$$

$$\bar{\mu}_{\mathbb{C}}([z, w]) = \sum_{i=1}^{d} z_i w_i e_i^*.$$

Differs from the toric case, the map $\bar{\mu}$ to $\mathfrak{n}^* \times \mathfrak{n}_{\mathbb{C}}^*$ is surjective, never with a bounded image.

For the purpose of this article, we always assume that $Y(\alpha, \beta)$ is a smooth manifold, i.e. $\{u_i\}$ is regular and (α, β) is regular value of the moment map μ(cf. [BD00] and [Kon08]). Parallel with previous subsection, we use hyperplane arrangement encoding the quotient construction. For the moment map takes value in $\mathfrak{m}^* \times \mathfrak{m}_{\mathbb{C}}^*$, the lift of (α, β) is $\Lambda = (\lambda^1, \lambda^2, \lambda^3)$, s.t.

$$\begin{cases} \alpha = \sum \lambda_i^1 \iota^* e_i^* \\ \beta = \sum (\lambda_i^2 + \sqrt{-1}\lambda_i^3) \iota^* e_i^* \end{cases}$$

Then we can construct the arrangement of codimension 3 flats (affine subspaces) in $\mathbb{R}^{3n}$,

$$H_i = H_i^1 \times H_i^2 \times H_i^3,$$

where

(2.1) $\quad H_i^h = \{x \in \mathfrak{m}^* | \langle u_i, x \rangle + \lambda_i^h = 0\}, \ (h = 1, 2, 3, \ i = 1, \dots, d)$

a prior with orientation u_i. For simplicity, we still denote it as $\mathcal{A}$. Vice versa, such a arrangement of 3-flats $\mathcal{A}$ determines a hyperkähler quotient $Y(\alpha, \beta)$.

We now investigate how the hyperkähler quotient $Y(\alpha, \beta)$ changes when the orientations are reversed. The original subtorus M is defined by the embedding $\iota\theta_k = a_k^i e_i$ and (α, β) has a lift Λ. Reversing the orientation by letting $\tilde{u}_j = \epsilon(j)u_j$, we get arrangement $\mathcal{A}_\epsilon$. The new subtorus $\tilde{M}$ is defined by the embedding $\tilde{\iota}\theta_k = \tilde{a}_k^i e_i$ where $\tilde{a}_k^j = \epsilon(j)a_k^j$ for $k = 1, \dots, m$, or equivalent saying $\tilde{\iota}^* e_j^* = \epsilon(j)\iota^* e_j^*$. This has a few

consequences. One is that the new subtorus $\tilde{M}$ acts on $\mathbb{H}^d$ is the same as the original subtorus M does on $\mathbb{H}^d$ by exchanging the positions of z_j and w_j if $\epsilon(j) = -1$. The second is about the new lift $\tilde{\Lambda}$. By Equation (2.1), we know that $\tilde{\Lambda}_j = \epsilon(j)\Lambda_j$, thus $\tilde{\alpha} = \sum \tilde{\lambda}_i^1 \tilde{\iota}^* e_i^* = \sum \lambda_i^1 \iota^* e_i^* = \alpha$, similarly $\tilde{\beta} = \beta$. Finally, the level set becomes

$$\frac{1}{2} \sum_{\epsilon(i)=1} (|z_i|^2 - |w_i|^2)\iota^* e_i^* + \frac{1}{2} \sum_{\epsilon(j)=-1} (|w_j|^2 - |-z_j|^2)\iota^* e_j^* = \alpha,$$

$$\sum_{\epsilon(i)=1} z_i w_i \iota^* e_i^* + \sum_{\epsilon(j)=-1} w_j(-z_j)\iota^* e_j^* = \beta.$$

Mapping (z_j, w_j) to $(w_j, -z_j)$, the level set is identical to the original one, so is the hyperkähler quotient. This means that toric hyperkähler manifolds according to the same arrangement with different orientations will be biholomorphic to each other, which is a significant difference from the toric variety(see a similar discussion in [Kon02], Lemma 4.2). Even though, to fix the position of variables, we still presume every arrangement is with given normal.

Example 2.3 (see [BD00]). *Let $\beta = 0$ and take α defined by the arrangement $u_1 = -f_1$, $u_2 = u_3 = f_1$ in $\mathfrak{n}^1$ where f_1 is the standard basis, and $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$, $\lambda_3 = 0$. The resulted toric hyperkähler manifold $Y(\alpha, 0)$ is the desingulariztion of $\mathbb{C}^2/\mathbb{Z}_3$(cf. [HS02], section 10).*
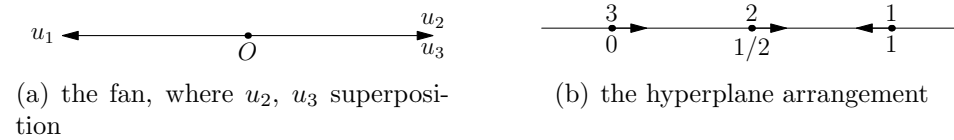

(a) the fan, where u_2, u_3 superposition

(b) the hyperplane arrangement

FIGURE 2. the fan and hyperplanes arrangement in Example 2.3

3. GIT DESCRIPTION AND CRITERIA FOR SEMI-STABILITY

3.1. **toric varieties.** Let us consider the GIT quotient of $\mathbb{C}^d$ by $M_{\mathbb{C}}$ with respect to the linearization induced by $\alpha \in \mathfrak{m}_{\mathbb{Z}}^*$. More explicitly, the element α induces the character $\chi_\alpha : M_{\mathbb{C}} \to \mathbb{C}^\times$, where $M_{\mathbb{C}}$ is the complexfication of M. Let $L^{\otimes m} = \mathbb{C}^d \times \mathbb{C}$ be the trivial holomorphic line bundle on which $M_{\mathbb{C}}$ acts by

$$((z), v)_m \zeta = ((z)\zeta, v\chi_\alpha(\zeta)^m)_m.$$

A point (z) is α-semi-stable if and only if there exists $m \in \mathbb{Z}_{>0}$ and a polynomial $f(p)$, where $p \in \mathbb{C}^d$, such that $f((p)\zeta) = f(p)\chi_\alpha(\zeta)^m$

7

for any $\zeta \in M_{\mathbb{C}}$ and $f(z) \neq 0$. We denote the set of α-semi-stable points in $\mathbb{C}^d$ by $(\mathbb{C}^d)^{\alpha-ss}$, then there is a categorical quotient ϕ : $(\mathbb{C}^d)^{\alpha-ss} \to (\mathbb{C}^d)^{\alpha-ss}//M_{\mathbb{C}}$, where $(\mathbb{C}^d)^{\alpha-ss}//M_{\mathbb{C}}$ is the GIT quotient of $\mathbb{C}^d$ by $M_{\mathbb{C}}$ respect to α, more precisely, the union of closed $M_{\mathbb{C}}$-orbits in $(\mathbb{C}^d)^{\alpha-ss}$(cf. [MFK94], and the readers are highly recommended to consult the lecture notes [Dol03] or [Tho06] if they prefer the variety rather than the abstract scheme setup). Sometimes $\mathbb{C}^d//_\alpha M_{\mathbb{C}}$ stands for this GIT quotient.

Unfortunately, the definition of stability respect to linearization is only effective when $\alpha \in \mathfrak{m}_{\mathbb{Z}}^*$, i.e. only corresponds to the algebraic toric variety with line bundle described by Newton ploytope with integer vertices. Following Konno's method in hyperkähler case, it can be generalized to any complex manifold.

Lemma 3.1. *Suppose that $\alpha \in \mathfrak{m}^*$,*
(1) A point $z \in \mathbb{C}^d$ is α-semi-stable if and only if

$$(3.1) \qquad\qquad \alpha \in \sum_{i=1}^{d} \mathbb{R}_{\geq 0} |z_i|^2 \iota^* e_i^*.$$

(2) Suppose $z \in (\mathbb{C}^d)^{\alpha-ss}$. Then the $M_{\mathbb{C}}$-orbit through z is closed in $(\mathbb{C}^d)^{\alpha-ss}$ if and only if

$$\alpha \in \sum_{i=1}^{d} \mathbb{R}_{>0} |z_i|^2 \iota^* e_i^*.$$

Proof. For convenience we give the proof of (1), and readers can find the essential proof of (2) in [Kon08]. Suppose $(z) \in (\mathbb{C}^d)^{\alpha-ss}$. Then there exists $m \in \mathbb{Z}_{>0}$ and a polynomial $f(p_1, \ldots, p_d)$ such that $f((p)\zeta) = f(p)\chi_\alpha(\zeta)^m$ and $f(z) \neq 0$. So we can select out a monomial $f_0(p) = \prod_{i=1}^{d} p_i^{a_i}$, where $a_i \in \mathbb{Z}_{>0}$, such that $f_0((p)\zeta) = f_0(p)\chi_\alpha(\zeta)^m$ and $f_0(z) \neq 0$. The second condition implies that $a_i = 0$ if $z_i = 0$. Moreover, the first condition implies $m\alpha = \sum_{i=1}^{d} a_i \iota^* e_i^*$. To see this, let θ_k be the standard basis of $\mathfrak{m}$ and $\rho \in \mathbb{C}^\times$, we have $\chi(\mathrm{Exp}(\rho\theta_k)) = e^{\rho\langle\alpha,\theta_k\rangle}$ and $(p)\mathrm{Exp}(\rho\theta_k) = (p_i e^{\rho\langle\iota\theta_k, e_i^*\rangle}) = (p_i e^{\rho\langle\iota^* e_i^*, \theta_k\rangle})$. Thus we proved Equation (3.1). $\qquad\square$

This definition of stability coincides the original GIT one when $\alpha \in \mathfrak{m}_{\mathbb{Z}}^*$. Followed by

Proposition 3.2. *(1)If we fix $\alpha \in \mathfrak{m}^*$, then the natural map σ : $X(\alpha) \to (\mathbb{C}^d)^{\alpha-ss}//M_{\mathbb{C}}$ is a homeomorphism.*
(2)If $\alpha \in \mathfrak{m}_{reg}^*$, then every $M_{\mathbb{C}}$-orbit is closed in $(\mathbb{C}^d)^{\alpha-ss}$. So the categorical quotient $(\mathbb{C}^d)^{\alpha-ss}//M_{\mathbb{C}}$ is a geometric quotient $(\mathbb{C}^d)^{\alpha-ss}/M_{\mathbb{C}}$.

Readers could consult [Kon08] for the proof. Thus we can identify the symplectic quotient $X(\alpha)$ with the GIT quotient $(\mathbb{C}^d)^{\alpha-ss}//M_{\mathbb{C}}$ in both algebraic and holomorphic case. This principle was established in [KN78], [MFK94] in the algebraic case. The general holomorphic version is proved in [Nak99].

In practice, a geometric interpretation of the stability is needed. Fix α, it is equivalent to give an arrangement $\mathcal{A}$. For any $z \in \mathbb{C}^d$, we associate a subregion of $\mathfrak{m}_+^*$. Let

$$F_z(i) = \begin{cases} H_i^{\geq 0} & if\, z_i \neq 0 \\ H_i & if\, z_i = 0 \end{cases},$$

where H_i and it's orientation come naturally from $\mathcal{A}$, denote $St_{\mathcal{A}}(z) = \bigcap_{i=1}^d F_z(i)$, which is a polytope or subpolytope. Via different lifts, $St_{\mathcal{A}}(z)$ only differs a parallel transport. And we leave the proof of Proposition 1.1 as a special case of toric hyperkähler manifold in next subsection, but let us add some remarks about it. Let J be a subset of $\{1, \cdots, d\}$, it is easy to see, a point $\{z|z_i = 0, \text{if } i \in J \text{ and } z_i \neq 0 \text{ if } i \in \bar{J}\}$ is semi-stable if and only if the hyperplanes $\{H_i | i \in J\}$ have nonempty intersection within the polytope Δ defined by $\mathcal{A}$. So our interpretation of stability can be viewed as a replacement of definition about $\mathbb{C}_\Delta^d$ of [Gui94a], [BD00] in the hyperplane arrangement context, and has a natural extension to the toric hyperkähler case.

Example 3.3. *Consider the hyperplanes arrangement of Fig 1(b), the $St_{\mathcal{A}}(z)$ for $z_2 = 0$ is the bottom of the trapezoid, and empty for $z_1 = z_3 = 0$, thus the first kind point is semi-stable and the second one is not semi-stable.*

3.2. **Toric hyperkähler manifold.** Similarly, consider the GIT quotient of $\mu_{\mathbb{C}}^{-1}(\beta)$ by $M_{\mathbb{C}}$ with respect to the linearization on the trivial line bundle induced by $\alpha \in \mathfrak{m}_{\mathbb{Z}}^*$. Denote the set of α-semi-stable points in $\mu_{\mathbb{C}}^{-1}(\beta)$ by $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}$, then there is a categorical quotient $\phi: \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss} \to \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}//M_{\mathbb{C}}$, where $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}//M_{\mathbb{C}}$ contains all closed $M_{\mathbb{C}}$ orbits in $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}$. Parallel with toric case, the stability condition can be generalized to any $\alpha \in \mathfrak{m}^*$(cf. [Kon08]).

Lemma 3.4. *Suppose that $\alpha \in \mathfrak{m}^*$,*
(1) A point $(z, w) \in \mu_{\mathbb{C}}^{-1}(\beta)$ is α-semi-stable if and only if

$$(3.2) \qquad \alpha \in \sum_{i=1}^d \mathbb{R}_{\geq 0}|z_i|^2 \iota^* e_i^* + \sum_{i=1}^d \mathbb{R}_{\geq 0}|w_i|^2(-\iota^* e_i^*).$$

(2) Suppose $(z, w) \in \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}$. Then the $M_{\mathbb{C}}$-orbit through (z, w) is closed in $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}$ if and only if

$$\alpha \in \sum_{i=1}^{d} \mathbb{R}_{>0} |z_i|^2 \iota^* e_i^* + \sum_{i=1}^{d} \mathbb{R}_{>0} |w_i|^2 (-\iota^* e_i^*).$$

Thus we can identify the symplectic quotient $Y(\alpha, \beta)$ with the GIT quotient $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}//M_{\mathbb{C}}$ for any $\alpha \in \mathfrak{m}^*$, and the geometric quotient $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-ss}/M_{\mathbb{C}}$ if (α, β) is a regular value.

It time to examine the numerical stability condition Equation (3.2) a little further. For α lies in $\mathfrak{m}^*$, there is

$$\alpha = \sum_{i=1}^{d} x_i \iota^* e_i^*.$$

Assume $\iota^* e_i^* = a_i^k \theta_k^*$, and $\alpha = \alpha^k \theta_k^*$, above equation turns to a linear equation system

$$Ax = \alpha,$$

where A is $m \times d$ matrix with entry a_i^k, and α represents the column vector $\{\alpha^k\}_{k=1}^m$. As we know α has lift λ s.t. $\alpha = \sum_{i=1}^{d} \lambda_i \iota^* e_i^*$ as a particular solution, so it is enough to consider the undetermined homogeneous system

$$Ax = 0.$$

If its n-dimensional solution space is denoted as $\mathfrak{N}$, then the solution of original inhomogeneous system is $\mathfrak{N}_\alpha \triangleq \lambda + \mathfrak{N}$, a n-plane in $\mathbb{R}^d$.

Proposition 3.5. *Regarding the point λ in $\mathfrak{N}_\alpha$ as the origin , projecting of $\mathfrak{N}_\alpha$ onto some standard $\mathbb{R}^n$, then we can identify $\pi_{\mathbb{R}^n}(\mathfrak{N}_\alpha)$ with $\mathfrak{n}^*$, and the hyperplane arrangement H_i is defined by $\pi_{\mathbb{R}^n}(\mathfrak{N}_\alpha \bigcap \{x_i = 0\})$, where $\{x_i = 0\}$ is the coordinate hyperplane in $\mathbb{R}^d$.*

Proof. Denote the standard basis of $\mathbb{R}^d$ as ∂_i. If $x \in \mathfrak{N}_\alpha \bigcap \{x_i = 0\}$ is a vector in $\mathbb{R}^d$, by definition, $\langle x, \partial_i \rangle = 0$. On another hand, x corresponds the vector $x - \lambda$ in $\mathfrak{N}_\alpha$, thus $\langle x - \lambda, \partial_i \rangle = \langle -\lambda, \partial_i \rangle = -\lambda_i$. Now $\mathfrak{N}_\alpha \bigcap \{x_i = 0\}$ looks a little different from the hyperplane arrangement in $\mathfrak{n}^*$(see Fig 3). We have to project it onto some $\mathbb{R}^n$. If $\partial_j \in \mathbb{R}^n$, then we simply have $u_j = \pi_{\mathbb{R}^n} \partial_j$, otherwise, the normal u_j will be the unique vector preserving above inner product which can be calculated by a little Euclidean geometry. Thus we recover the arrangement, and it is easy to see that the resulted arrangement to different projections differ at most $GL(n, \mathbb{Z})$ transforms, i.e. all equivalent. $\square$

Example 3.6. *Consider the 1-dimensional torus acts on $\mathbb{H}^3$ diagonally, and choose $\alpha = 3$ with lift $\lambda = (1, 1, 1)$, we have a two dimensional solution space and its projection onto $\mathfrak{n}^* \cong \mathbb{R}^2$ illustrated in Fig 3, where O is the projection of λ.*

(a) the solution space $\mathfrak{N}_\alpha$

(b) projected onto $x_1 x_2$ plane identifies with the arrangement in $\mathfrak{n}^*$
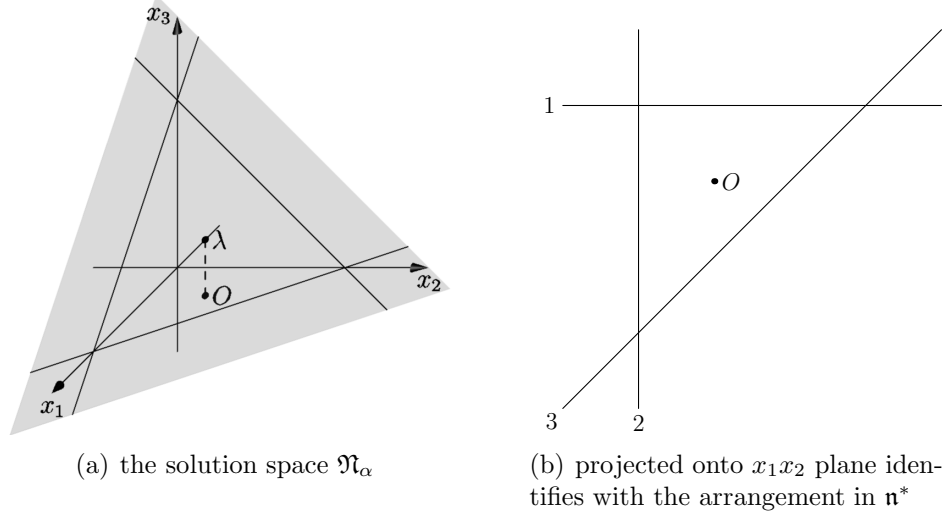
FIGURE 3. the relation of solution space and hyperplane arrangements

Notice that the projection $\pi_{\mathbb{R}^n}$ does not affect the intersection relations of the hyperplane in $\mathfrak{N}_\alpha$ and $\mathfrak{n}^*$, i.e. the relative positions. So we will not distinguish them from each other in the later practice. In effect, the $\mathfrak{N}_\alpha$ picture is more close to the intrinsic geometry of toric variety and toric hyperkähler manifold.

As promised, we give the geometric interpretation of the stability condition. Recall Equation (3.2), we easily have

Lemma 3.7. *Given I, J subset of $\{1, \ldots, d\}$, the point $\{(z, w)|z_i = 0, w_j = 0, if\ and\ only\ if\ i \in I, j \in J\}$ is α-semi-stable if and only if there exists a solution $x \in \mathfrak{N}_\alpha$ s.t. $x_i \leq 0, x_j \geq 0, for\ i \in I, j \in J$.*

Fixing the arrangement $\mathcal{A}$, we will associate a region to every point $(z, w) \in \mathbb{H}^d$. Set

$$F_z(i) = \begin{cases} H_i^{\geq 0} & if\ z_i \neq 0 \\ H_i & if\ z_i = 0 \end{cases},$$

and

$$F_w(i) = \begin{cases} H_i^{\leq 0} & if\ w_i \neq 0 \\ H_i & if\ w_i = 0 \end{cases},$$

11

define $St_{\mathcal{A}}(z,w) = \bigcap_{i=1}^{d}(F_z(i)\bigcup F_w(i))$, which is a union of polytopes or subpolytopes. Then we are in the position to prove Proposition 1.2.

Proof. We already know the hyperplane arrangement is the solution space cut by the coordinate hyperplanes in $\mathbb{R}^d$. The non empty of $St_{\mathcal{A}}(z,w)$ of the point $\{(z,w)|z_i = 0, w_j = 0, \text{ if and only if } i \in I, j \in J\}$ implies that there is a $x \in \mathfrak{N}_\alpha$ s.t. $x_i \leq 0, x_j \geq 0, for\ i \in I, j \in J$. By Lemma 3.7, This means this point is semi-stable, and vice versa. $\square$

Proposition 1.1 concerning the toric case is an easy consequence by letting all w_i be zero.

4. Coverings of toric hyperkähler manifolds

In this part, β is taken to be zero. It is enough to merely consider the hyperplanes arrangement H^1. We will abuse the notation using H in stead of H^1.

4.1. The geometry of the extended core and core. The subset of $Y(\alpha, 0)$

$$Z = \bar{\mu}_{\mathbb{C}}^{-1}(0) = \{[z,w] \in Y(\alpha, 0)|z_i w_i = 0 \text{ for all } i\},$$

is called the extended core by Proudfoot(cf. [Pro04]), which naturally breaks into components

$$Z_\epsilon = \{[z,w] \in Y(\alpha, 0)|w_i = 0 \ if\ \epsilon(i) = 1\ and\ z_i = 0\ if\ \epsilon(i) = -1\}.$$

The variety $Z_\epsilon \subset Y(\alpha, 0)$ is a n-dimensional isotropy Kahler subvariety of $Y(\alpha, 0)$ with an effective hamiltonian T^n-action, hence a toric variety itself. It is not hard to see this is just the toric variety corresponding to the oriented hyperplane arrangement $\mathcal{A}_\epsilon$. Denote the associated polytope of $\mathcal{A}_\epsilon$ as Δ_ϵ. The set $Z_{cpt} = \bigcup_{\epsilon \in \Theta_{cpt}} Z_\epsilon$, where $\Theta_{cpt} = \{\epsilon|\Delta_\epsilon \text{ bounded}\}$, is called the core, union of compact toric varieties $X(\mathcal{A}_\epsilon), \epsilon \in \Theta_{cpt}$.

It is natural to ask when a toric hyperkähler manifold has nonempty core. Let $\mathcal{A}$ be the smooth arrangement and u_i its normals, it amounts to check wether $\mathcal{A}$ contains bounded n-dimensional polytopes. Then $\mathcal{A}$ does not contain bounded polytopes if and only if there is a subset K of $\{1, \ldots, d\}$, the length $|K| < n$, s.t. for each $k \in K$, u_k is independent of all others $\{u_i|i \in \{1, \ldots, d\}, i \neq k\}$. In this case, $Y(\alpha, 0)$ can be written as $\tilde{Y}(\tilde{\alpha}, 0) \times \mathbb{H}^{|K|}$, where $\tilde{Y}(\tilde{\alpha}, 0)$ is a $4(n - |K|)$ dimensional toric hyperkähler manifold(see Fig 4(a)). Thus except the trivial product case where some u_i is linear independent of all others, the toric hyperkähler variety $Y(\alpha, 0)$ has nonempty core, which will be take for granted in the remaining part.
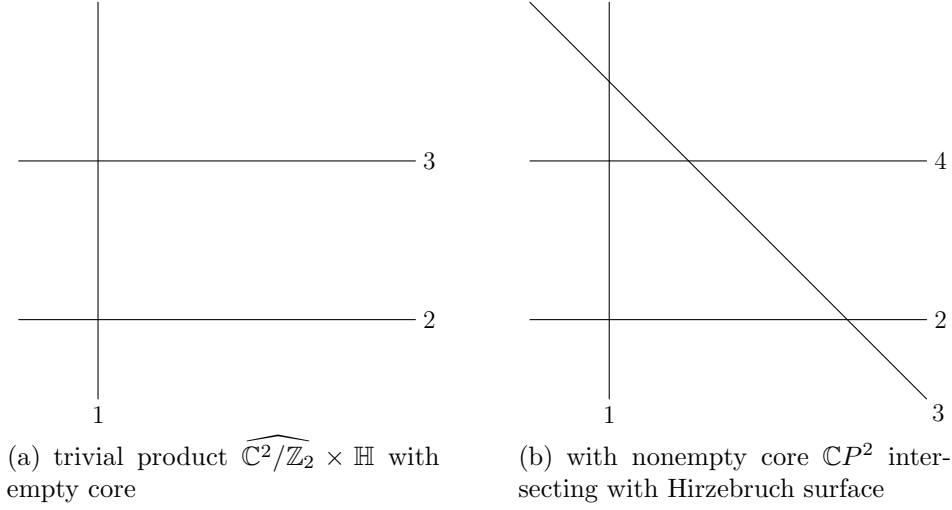
12

(a) trivial product $\widehat{\mathbb{C}^2/\mathbb{Z}_2} \times \mathbb{H}$ with empty core

(b) with nonempty core $\mathbb{C}P^2$ intersecting with Hirzebruch surface

FIGURE 4. bounded polytopes in hyperplane arrangements

4.2. **Cotangent bundle of toric variety.** Another important result in [BD00] concerns the cotangent bundle of toric variety. Suppose $\mathcal{A}$ is nonempty, the toric variety $X(\mathcal{A})$ is automatically in the core of $Y(\alpha, 0)$. Dancer and Bielawski proved that $T^*X(\mathcal{A})$ isomorphic to an open subset U. Later, this result was generalized by Konno(see [Kon02], lemma 4.2), namely

Lemma 4.1. *Let $Y(\alpha, 0)$ be a toric hyperkähler manifold. If $X(\mathcal{A}_\epsilon)$ is not empty, then its cotangent bundle $T^*X(\mathcal{A}_\epsilon)$ is contained in $Y(\alpha, 0)$ as an open subset. Moreover, the hyperkähler metric restricted on the zero section of this cotangent bundle, is the canonical metric on toric variety $X(\mathcal{A}_\epsilon)$.*

From now on, the open set isomorphic to $T^*X(\mathcal{A}_\epsilon)$ is denoted as U_ϵ(stands for $U_{\mathcal{A}_\epsilon}$ in effect). We will only sketch the proof in case $\epsilon(i) = 1$ for all i, and reader could consult [Kon02] for the detail. Consider the open subset $(\mathbb{C}^d)^{\alpha-ss} \times \mathbb{C}^d$ of $\mathbb{H}^d$. The group M acts on it freely, so we can perform the hyperkähler quotient construction on $(\mathbb{C}^d)^{\alpha-ss} \times \mathbb{C}^d$ and obtain an open subset U of $Y(\alpha, 0)$. In order to show that U is isomorphic to $T^*X(\mathcal{A})$, we identify U with the GIT quotient $((\mathbb{C}^d)^{\alpha-ss} \times \mathbb{C}^d) \cap \mu_{\mathbb{C}}^{-1}(0)/M_{\mathbb{C}}$, i.e.

$$\{(z, w) \in (\mathbb{C}^d)^{\alpha-ss} \times \mathbb{C}^d | \sum_{i=1}^{d} z_i w_i \iota^* e_i^* = 0\}/M_{\mathbb{C}},$$

13

where we directly apply geometric quotient for $(\alpha, 0)$ is a regular value. Above equation simply says that the vector $w \in T_z^*(\mathbb{C}^d)^{\alpha-ss}$ annihilates the tangent vectors along the $M_\mathbb{C}$ orbits, i.e. the vertical tangent vectors of the projection $(\mathbb{C}^d)^{\alpha-ss} \to X(\mathcal{A})$. This show that U is isomorphic to $T^*X(\mathcal{A})$.

4.3. **The proof of the main theorem.** All the nonempty canonical open sets U_ϵ naturally constitute a covering of $Y(\alpha, 0)$. Bielawski pointed it out that in [Bie99], the toric hyperkähler manifold can be viewed as the gluing the cotangent bundle of some toric varieties. So it means that we can choose all the toric varieties in the extended core to do this job, but in a subtle way. The only chance to see how these cotangent bundles are glued is when there is as few as components are involved, i.e. using few cotangent bundles to glue the toric hyperkähler manifold. In the following, we will see compact toric varieties in the core play such a role.

Theorem 4.2. *Let $Y(\alpha, 0)$ be a smooth toric hyperkähler manifold with nonempty core, then the canonical open set $U_\epsilon \cong T^*X(\mathcal{A}_\epsilon), \epsilon \in \Theta_{cpt}$ is a covering of $Y(\alpha, 0)$, i.e. the cotangent bundles of compact toric varieties in the core are enough to glue $Y(\alpha, 0)$.*

Proof. We first check the stability condition with respect to the toric variety $X(\mathcal{A}_\epsilon), \epsilon \in \Theta_{cpt}$. The point $(z_J, w_{\bar{J}}) \in \mathbb{C}^d$, where $J = \{i | \epsilon(i) = 1\}$ and $\bar{J}$ is its complement, is semi-stable if and only if

(4.1) $$\alpha \in \sum_J \mathbb{R}_{\geq 0} |z_i|^2 \iota^* e_i^* + \sum_{\bar{J}} \mathbb{R}_{\geq 0} |w_i|^2 (-\iota^* e_i^*).$$

If we denote these points by $(\mathbb{C}^d)_\epsilon^{\alpha-ss}$, then $(\mathbb{C}^d)_\epsilon^{\alpha-ss} \times \mathbb{C}^d$ is a subset of $(\mathbb{H}^d)^{\alpha-ss}$. Performing the hyperkähler quotient on it results in U_ϵ, which isomorphic to $T^*X(\mathcal{A}_\epsilon)$. To prove the union of U_ϵ covers $Y(\alpha, 0)$, it suffices to show that the union of $(\mathbb{C}^d)_\epsilon^{\alpha-ss} \times \mathbb{C}^d$ covers $(\mathbb{H}^d)^{\alpha-ss}$. In practice, we show that if a point lies out side for every $(\mathbb{C}^d)_\epsilon^{\alpha-ss} \times \mathbb{C}^d$, then it must be unstable.

We claim that if (z, w) lies out side of $(\mathbb{C}^d)_\epsilon^{\alpha-ss} \times \mathbb{C}^d$ for some fixed $\epsilon \in \Theta_{cpt}$, then $St_\mathcal{A}(z, w) \cap \Delta_\epsilon = \emptyset$. Without losing any generality, we assume $\epsilon(i) = 1$. The point $\{z | z_i = 0, \text{if and only if } i \in I\}$ is unstable for $X(\mathcal{A})$ means that $\{H_i\}_{i \in I}$ do not have intersection in Δ. Since $z_i = 0$ for $i \in I$, the region $St_\mathcal{A}(z, w)$ now lies in the intersection of half spaces $\{H_i^{0\leq}\}_{i \in I}$, $St_\mathcal{A}(z, w)$ can not contact with Δ. If not, let x be the common intersection of Δ and $St_\mathcal{A}(z, w)$, then $x \in H_i^{\geq 0}$ and $x \in H_i^{\leq 0}$, for all the $i \in I$. It means that $x \in \bigcap_{i \in I} H_i$ is a common intersection of H_i, contradiction with z is unstable with respect to the toric variety.

Same argument applies for any $\epsilon \in \Theta_{cpt}$, so $St_\mathcal{A}(z,w)$ are not adjacent to any Δ_ϵ. While Δ_ϵ are all the bounded polytopes and every unbounded polytope and its subpolytopes must have intersection with one of them(for example, see Fig 4(b)), thus the only possibility is $St_\mathcal{A}(z,w)$ is empty, i.e. (z,w) is unstable. $\qquad\square$

This result is not surprising. By the construction of the open set $U_\epsilon \cong T^*X(\mathcal{A}_\epsilon)$ from GIT method, we have in effect

Corollary 4.3. *Let $Y(\alpha,0)$ be a toric hyperkähler manifold. If $X(\mathcal{A}_\epsilon)$ is not empty, then its cotangent bundle $T^*X(\mathcal{A}_\epsilon)$ is contained in $Y(\alpha,0)$ as an open dense subset.*

Proof. Check the semi-stable condition Equation (4.1), there are only two possibilities. One is the set $(\mathbb{C}^d)^{\alpha-ss}_\epsilon$ is empty. Otherwise, if there is some point $(z_J, w_{\bar{J}})$ satisfying the condition, then any $\{(z_J, w_{\bar{J}})|z_i \neq 0 \text{ for } \epsilon(i) = 1 \text{ and } w_i \neq 0 \text{ for } \epsilon(i) = -1\}$ belong to $(\mathbb{C}^d)^{\alpha-ss}_\epsilon$, which means $(\mathbb{C}^d)^{\alpha-ss}_\epsilon$ is dense in $\mathbb{C}^d$. Then the corollary follows from that $\mu_\mathbb{C}^{-1}(0) \bigcap (\mathbb{C}^d)^{\alpha-ss}_\epsilon \times \mathbb{C}^d$ is dense in $\mu_\mathbb{C}^{-1}(0)^{\alpha-ss}$ and on which the complex torus $M_\mathbb{C}$ acts effectively. $\qquad\square$

Further more, we can verify that the following conjecture holds in most low dimensional cases

Conjecture 4.4. *The complement of each open dense set $U_\epsilon \cong T^*X(\mathcal{A}_\epsilon)$ in the toric hyperkähler manifold $Y(\alpha,0)$ is of complex codimension n and constituted by several toric varieties in the extended core of $Y(\alpha,0)$.*

The method developed in the proof of the main theorem is not effective in this problem probably because that the state set $St_\mathcal{A}(z,w)$ is not adjacent to some Δ_ϵ is much weaker than $St_\mathcal{A}(z,w)$ is empty itself. So it is hopeful utilizing deep combinatorial theory to prove this conjecture.

References

[BD00]   R. Bielawski and A. Dancer. The geometry and topology of toric hyperkähler manifolds. *Commun. Anal. Geom.*, 8:726–760, 2000.

[Bie99]   R. Bielawski. Complete hyperkähler $4n$-manifolds with n commuting tri-hamiltonian vector fields. *Math. Ann.*, 314:505–528, 1999.

[Del88]   T. Delzant. Hamiltioniens periodiques et images convexe de l'application moment. *Bull. Soc. Math. France*, 116:315–339, 1988.

[DH98]   I. Dolgachev and Y. Hu. Variations of geometric invariant theory quotients. *Publ.Math.de l'IHES*, 87:5–51, 1998.

[Dol03]   I. Dolgachev. *Lectures on Invariant Theory.* Cambridge Uni. Press, 2003.

[Ful93]   W. Fulton. *Introduction to Toric Varieties.* Princeton Uni. Press, 1993.

[Gui94a]   V. Guillemin. Kahler structures on toric varieties. *J. Diff. Geom.*, 40:285–309, 1994.

[Gui94b]   V. Guillemin. *Moment Maps and Combinatorial Invariants of Hamiltonian T^N-spaces*. Birkhauser, 1994.

[HKLR87]   N. Hitchin, A. Karlhede, U. Lindstrom, and M. Rocek. Hyperkähler metrics and supersymmetry. *Commun. Math. Phys.*, 108:535–589, 1987.

[HS02]   T. Hausel and B. Sturmfels. Toric hyperkähler varieties. *Documenta Mathematica*, 7:495–534, 2002.

[KN78]   G. Kempf and L. Ness. *On the length of vectors in representation spaces*, volume 732 of *Lecture Notes in Math.* 1978.

[Kon02]   H. Konno. *The Topology of Toric Hyperkähler manifolds*, volume 34 of *Advanced Studies in Pure Mathematics*. 2002.

[Kon03]   H. Konno. Variation of toric hyperkähler manifolds. *International Journal of Mathematics*, 14:289–311, 2003.

[Kon08]   H. Konno. The geometry of toric hyperkähler varieties. In *Toric Topology, Contemp. Math. 460*, pages 241–260, Osaka, 2008.

[MFK94]   D. Mumford, J. Forgaty, and F. Kirwan. *Geometric Invariant Theory*. Springer, 1994.

[Nak99]   H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*. American Math. Soc., 1999.

[Oda88]   T. Oda. *Convex Bodies and Algebraic GeometryAn Introduction to the Theory of Toric Varieties*, volume 3 of *Ergebnisse der Math.* Springer-Verlag, 1988.

[Pro04]   N. Proudfoot. *Hyperkähler Analogues of Kahler Quotients*. PhD thesis, University of California, Berkeley, 2004.

[Tho06]   R. Thomas. Notes on GIT and symplectic reduction for bundles and varieties. Arxiv preprint math/0512411, 2006.

Craig van Coevering: craigvan@ustc.edu.cn

Wei Zhang: zhangw81@ustc.edu.cn

School of Mathematical Sciences
University of Science and Technology of China
Hefei, 230026, P. R.China